Scattering matrix descriptors of Trefftz finite elements

Yuriy Shlepnev, shlepnev@ieee.org
Mentor Graphics Corporation

Abstract
This paper explains some details related with the choice of the surface or mortar basis functions for the Trefftz finite elements. It is shown that appropriate normalization of the surface basis functions lead to unambiguous definition of the generalized admittance matrix descriptor of the element and to a possibility to define scattering matrix descriptor of the element. A brick element is investigated to illustrate the approach.

Introduction

The Trefftz finite elements have been recently introduced into computational electromagnetics [1]-[5]. The procedure of the element building is relatively simple. Solutions of the homogeneous Maxwell's equations are used as the basis functions to expand the interior element fields into Fourier series. Finite number of plane waves propagating in different directions may be chosen as such basis functions. To build a descriptor of the element, an additional set of basis functions has to be defined on the surface of the element. The purpose of the surface basis functions is to control energy transmission through the surface of the element. Projections of the interior fields on the surface basis functions link the interior expansion with the exterior basis functions and finally lead to admittance or scattering matrix descriptors of the element. To control the energy transmission through the element surface, the surface functions have to satisfy some orthogonality conditions. This was not covered in the papers [1]-[5] and can be ambiguous even in the case of a brick for example. Only Trefftz elements built with Galerkin's projectors are considered in this paper because of they are conservative by construction.

Trefftz element surface field expansion

Let's consider a polytope element $\Omega \in \mathbb{R}^3$ with surface S and N_f number of faces. Power transferred through the surface of the element can be expressed through components of the electric and magnetic fields tangential to the element surface:

$$P = \oint_S \left(\overline{E}_{St} \times \overline{H}^*_{St} \right) \cdot \overline{n}_S ds \tag{1}$$

where $\overline{n}_S$ is a normal vector directed inside the element, and $\overline{E}_{St}$ and $\overline{H}_{St}$ are components of the element interior field defined on the surface of the element and tangential to the surface. $\overline{E}_{St}$ and $\overline{H}_{St}$ are expanded into Fourier series inside the element according to procedure described in [1]-[5]. To expand element surface field into a Fourier series, a finite set of vector functions is used:

$$f_{l(m)} = \begin{pmatrix} \overline{e}_{l(m)} \\ \overline{h}_{l(m)} \end{pmatrix}, \quad l = 1,...N_f, \; m = 1, M_l \tag{2}$$

where M_l is the number of basis functions at the face number l, $\overline{n}_l$ is the normal vector to the face number l directed inside the element, $\overline{e}_{l(m)}$ is electric field component and $\overline{h}_{l(m)}$ is magnetic

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **23 APR 2004**	2. REPORT TYPE **N/A**	3. DATES COVERED **-**
4. TITLE AND SUBTITLE **Scattering matrix descriptors of Trefftz finite elements**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Mentor Graphics Corporation**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001763, Annual Review of Progress in Applied Computational Electromagnetics (20th) Held in Syracuse, NY on 19-23 April 2004., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**8**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

field component of the basis function. Minimal number of the basis functions per face for a 3-dimensional problem is two, to account two orthogonal polarizations of electric field on the face. For a problem with just one polarization of electric field, one basis function per face is the minimal number. To construct conservative Trefftz finite elements (TFE), it is necessary to control energy transmission through the surface of the element. In other words, the vector basis functions (2) have to be an orthogonal set with a power scalar product defined on the surface of the element as follows:

$$\left(f_{l(m)}, f_{k(n)} \right) = \oint_S \left(\bar{e}_{l(m)} \times \bar{h}^*_{k(n)} \right) \cdot \bar{n}_S ds = \delta_{l(m),k(n)} \tag{3}$$

where $\delta_{l(m),k(n)}$ is Kronecker symbol equal either to the unit if $l = k$ and $m = n$ or to zero otherwise. If condition (3) is satisfied, the electric and magnetic fields on the surface of the element can be expanded into Fourier series:

$$\begin{aligned} \bar{E}_{St} &= \sum_{l=1}^{N_f} \sum_{m=1}^{M_l} v_{l(m)} \bar{e}_{l(m)} \\ \bar{H}_{St} &= \sum_{l=1}^{N_f} \sum_{m=1}^{M_l} i_{l(m)} \bar{h}_{l(m)} \end{aligned} \tag{4}$$

The coefficients of expansion $v_{l(m)}$ and $i_{l(m)}$ can be found as follows:

$$\begin{aligned} v_{l(m)} &= \int_{Sl} \left(\bar{E}_{St} \times \bar{h}^*_{l(m)} \right) \cdot \bar{n}_l ds = \int_{Sl} \bar{E}_{St} \cdot \bar{e}^*_{l(m)} ds \\ i_{l(m)} &= \int_{Sl} \left(\bar{e}^*_{l(m)} \times \bar{H}_{St} \right) \cdot \bar{n}_l ds = \int_{Sl} \bar{H}_{St} \cdot \bar{h}^*_{l(m)} ds \end{aligned} \tag{5}$$

With the normalization (3), power transferred through the surface of the element (1) can be expressed as follows:

$$P = \sum_{l}^{N_f} \sum_{m}^{M_l} v_{l(m)} i^*_{l(m)} \tag{6}$$

This definition of power corresponds to the well-known circuit theory definition. Thus, an admittance or impedance descriptor relating $v_{l(m)}$ and $i_{l(m)}$ defines a multiport with voltages $v_{l(m)}$ and currents $i_{l(m)}$ at the ports. There is one condition that has to be satisfied to obtain Fourier series expansion (4) with the simple multiport power definition (6). This is a possibility to build a set of orthogonal basis functions (3) on the surface of the element. How to proceed with such construction is not obvious for a polytope in a general case. It can be done rigorously only for rectangular or brick elements that have rectangular faces and adjoined faces belong to the orthogonal planes. Thus, the rest of this paper is devoted to the brick elements only.

<u>Basis functions for a rectangular face</u>

Minimal number of the basis functions per rectangular face for a 3-diminsional problem is two, to account two polarizations of electric field. Each basis function contains one constant vector component of electric field and one constant vector component of magnetic field. The basis functions for a face l are shown in Fig. 1. Though the choice of the directions of the bases is obvious, it is not so with the magnitudes. We can randomly select magnitude of the electric field and define magnetic field component accounting normalization condition (3). In general, the basis functions may be expresses as follows:

$$
\begin{aligned}
&\overline{e}_{l(1)}=\alpha_1\overline{x}_1,\ \ \overline{h}_{l(1)}=\frac{\beta_1}{Z_0}\left(\overline{n}_l\times\overline{e}_{l(1)}\right)=\beta_1\frac{\overline{x}_2}{Z_0}\\
&\overline{e}_{l(2)}=\alpha_2\overline{x}_2,\ \ \overline{h}_{l(2)}=\frac{\beta_2}{Z_0}\left(\overline{n}_l\times\overline{e}_{l(2)}\right)=-\beta_2\frac{\overline{x}_1}{Z_0}
\end{aligned}
\tag{7}
$$

where $Z_0=\sqrt{\mu/\varepsilon}$ is characteristic impedance of the medium. There are multiple possibilities to define $\alpha_i,\beta_i,\ i=1,2$, but two choices correspond to some physical situations. The first choice is

$$
\begin{aligned}
&\alpha_1=\frac{\eta}{\Delta 2},\ \ \beta_1=\frac{\eta}{\Delta 1}\\
&\alpha_2=\frac{\eta}{\Delta 1},\ \ \beta_2=\frac{\eta}{\Delta 2}
\end{aligned}
\tag{8}
$$

where coefficient η has to be chosen to satisfy the unit power normalization condition (3):

$$\eta=\sqrt{Z_0} \tag{9}$$

The ratio of magnitudes of the electric and magnetic fields is independent of η:

$$Z_{l(1)}=\frac{\left|\overline{e}_{l(1)}\right|}{\left|\overline{h}_{l(1)}\right|}=\frac{\Delta 1}{\Delta 2}Z_0,\ \ Z_{l(2)}=\frac{\left|\overline{e}_{l(2)}\right|}{\left|\overline{h}_{l(2)}\right|}=\frac{\Delta 2}{\Delta 1}Z_0 \tag{10}$$

$Z_{l(1)}$ and $Z_{l(2)}$ are the characteristic impedances of the parallel-plate waveguides with electric walls shown by thick black lines and with magnetic walls shown by thin dashed lines in Fig. 1. With the choice (8), $v_{l(m)}$ and $i_{l(m)}$ define normalized voltage and current of the parallel-plate waveguides. With $\eta=1$ the admittance descriptor is going to be regular admittance matrix. Another choice of the coefficients in the basis functions (7) is the following:

$$
\begin{aligned}
&\alpha_1=\beta_1=\frac{\eta}{\sqrt{\Delta 1\Delta 2}}\\
&\alpha_2=\beta_2=\frac{\eta}{\sqrt{\Delta 1\Delta 2}}
\end{aligned}
\tag{11}
$$

The unit power transfer normalization condition (3) requires the same value (9) for the coefficient η. The ration of the magnitudes of electric and magnetic components of the basis functions with (11) is

$$Z_{l(1)}=\frac{\left|\overline{e}_{l(1)}\right|}{\left|\overline{h}_{l(1)}\right|}=Z_0,\ \ Z_{l(2)}=\frac{\left|\overline{e}_{l(2)}\right|}{\left|\overline{h}_{l(2)}\right|}=Z_0 \tag{12}$$

This definition corresponds to the basis functions defined as the TEM waves. A normalized admittance matrix obtained with either definitions of the surface basis functions relates surface expansion coefficients of electric and magnetic field of the basis functions as follows:

$$\tilde{i}=Y_{nr}\cdot\tilde{v},\quad \tilde{i},\tilde{v}\in C^N,\ Y_{nr}\in C^{N\times N} \tag{13}$$

where $\tilde{v}$ and $\tilde{i}$ are vectors composed of unknown surface expansion coefficients (4), N is the total number of the element surface basis functions. The normalized admittance matrix can be formally transformed into a scattering matrix. Incident and reflected waves can be introduced formally as

$$\tilde{c}^{+} = \tfrac{1}{2}\left(\tilde{v} + \tilde{i}\right), \quad \tilde{c}^{-} = \tfrac{1}{2}\left(\tilde{v} - \tilde{i}\right), \quad \tilde{c}^{+}, \tilde{c}^{-} \in C^{N} \tag{14}$$

Corresponding scattering matrix can be defined as

$$\tilde{c}^{-} = S \cdot \tilde{c}^{+}, \qquad S \in C^{N \times N} \tag{15}$$

It can be expressed through the admittance matrix as

$$S = (U - Y_{nr}) \cdot (U + Y_{nr})^{-1} \tag{16}$$

where U is the unit matrix N by N. The scattering matrix obtained in such way is a generalized scattering matrix and does not depend on normalization of the bases.

<u>Brick element descriptors</u>

Now it is easy to show, that with both definitions of the basis functions (7) with either coefficients (8) and (9) or (11) and (9), we can obtain exactly the same normalized admittance matrix of a brick element for example. The admittance matrix of a brick may be found either with formulas for its elements published in [2] or with more stable generalized matrix procedure described in [4], [5]. The electric field components in [2] have been chosen as constant unit vector functions defined on the faces of the brick. Magnetic field components of the basis functions in [2] have been also chosen to be constant unit vector functions defined through the electric field components. With the projectors defined in [2], the resulting admittance matrix has to be normalized in general case to be converted into scattering matrix. Such normalization is not required in case of the cubic element that is investigated in the paper [2]. For a brick element the choice of basis functions and projectors of paper [2] require additional normalization of the admittance matrix before the conversion to the scattering descriptor. The admittance matrix Y defined by expression (24) of paper [2] can be transformed into the normalized admittance matrix (13) as follows:

$$Y_{nr} = Z_0 \Theta \cdot Y \cdot \Theta^{-1} \tag{17}$$

where Θ is a diagonal matrix defined as follows:

$$\Theta = \begin{pmatrix}
\sqrt{\Delta z \Delta y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\Delta z \Delta y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\Delta z \Delta y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\Delta z \Delta y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\Delta z \Delta x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\Delta z \Delta x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta z \Delta x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta z \Delta x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta x \Delta y} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta x \Delta y} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta x \Delta y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\Delta x \Delta y}
\end{pmatrix} \tag{18}$$

Matrix Y_{nr} obtained with (17) is a generalized admittance matrix descriptor of the element and does not depend on the choice of the basis functions. It is interesting to investigate the un-normalized admittance matrix with the two different sets of basis functions to be able to correlate it with the actual values of electric and magnetic fields. With the parallel-plate waveguide definition (8), the un-normalized matrix is

$$Y_{ppw} = \Theta_h \cdot Y \cdot \Theta_e^{-1} \tag{19}$$

where

$$\Theta_e = \begin{pmatrix} \Delta z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta y \end{pmatrix} \qquad \Theta_h = \begin{pmatrix} \Delta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta x \end{pmatrix} \tag{20}$$

Admittance matrix Y_{ppw} relates voltage and current according to their regular definitions. Wave channels of multiport corresponding to brick have characteristic impedances defined as

$$Z_c = Z_0 \Theta_e \cdot \Theta_h^{-1} \tag{21}$$

Normalization of the matrix Y_{ppw} to the characteristic impedances of the channels (21) $Y_{nr} = \sqrt{Z_c} \cdot Y_{ppw} \cdot \sqrt{Z_c}$ gives the generalized admittance matrix Y_{nr} defined by (17). Admittance matrix Y_{ppw} can be also normalized to characteristic impedance of medium Z_0 or to 50 Ohm before transformation to the scattering matrix (16). It does not mater to what value it is normalized as long as it is appropriately accounted at the matrix re-composition stage or at the interface between two matrices. With the TEM waves definition (11), the un-normalized admittance matrix is

$$Y_{tem} = \Theta \cdot Y \cdot \Theta^{-1} \tag{22}$$

This matrix relates directly coefficients of expansion of electric end magnetic fields. Characteristic impedances of the wave channels in that case coincide with the characteristic impedance of the medium Z_0, thus normalization to Z_0 produces the generalized admittance matrix Y_{nr}. Note, that all descriptors are technically equivalent. The normalization does not change the physics of the problem and is important only in context of the problem at the element assembly stage or to construct an appropriate energy transition procedure between the elements in time domain for example.

Conservativeness of the brick element

To illustrate the conservative properties of the brick Trefftz finite elements, let us consider a cubic element with $\Delta x = \Delta y = \Delta z = 0.02\,m$ filled with vacuum. The element size is approximately equal to wavelength at 15 GHz. Formulas for elements of admittance matrix from [2] have been used to compute element descriptor. The admittance descriptor is converted into generalized scattering matrix according to procedure described in this paper. For a cubic element built with the collocation projector, the scattering matrix has only one distinct element with non-zero constant magnitude 0.5 and linear phase $-0.5k\Delta$. Here k is the wave propagation constant and Δ is the element size. For a cubic element built with Galerkin projectors, there are just two distinct non-zero elements in the scattering matrix descriptor [2]. Frequency dependencies of their magnitudes and phases are plotted in Fig. 2 and Fig. 3. Note, that magnitudes are frequency dependent in that case and phases are slightly non-linear. Both cubic elements have unitary scattering matrix descriptor at all frequencies that can be proved analytically. Now, let's build the generalized scattering descriptors for a brick element with $\Delta x = 0.01m$, $\Delta y = \Delta z = 0.02\,m$ and filled with vacuum. Frequency dependencies of their magnitudes and phases for a brick element

built with the Galerkin projectors are plotted in Fig. 4 and Fig. 5. The magnitudes values are asymptotically approaching to the static values that can be calculated from connection of parallel-plate waveguides. If scattering matrix is not generalized the asymptotic is going to be different and consistent with the normalization of the wave channels. The scattering matrix is unitary in that case as in the case of cubic element. Note that scattering matrix descriptor of the brick element built with the collocation projector holds this property only approximately if element size is substantially smaller than wavelength. This is because of the element construction procedure does not guarantee the conservativeness with collocation projectors. On the other hand the Galerkin projectors with the orthogonal set of the element surface basis functions allows to enforce the conservativeness of the element descriptor.

Acknowledgement

The author gratefully acknowledges Vladimir Podlozniy and Izuru Kanno for helpful discussions.

References

1. Y.O. Shlepnev, Trefftz-type brick finite elements for electromagnetics. – In proceedings of the 17th Annual Review of Progress in Applied Computational Electromagnetics (ACES'2001), Monterey, CA, March 19-23, 2001, pp.345-352.
2. Y.O. Shlepnev, Trefftz finite elements for electromagnetics. - *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-50, pp. 1328-1339, May, 2002.
3. Y.O. Shlepnev, Polygonal Trefftz finite elements for electromagnetics. – In proceedings of the 18th Annual Review of Progress in Applied Computational Electromagnetics (ACES'2002), Monterey, CA, March 17-22, 2002, corrected version at www.shlepnev.com.
4. Y.O. Shlepnev, Building Trefftz finite elements for electromagnetic problems, - Mathematical Methods in Electromagnetic Theory (MMET'2002), Kiev, Ukrain, September 2002.
5. Y.O. Shlepnev, T-elements for 3-D electromagnetic problems. – In proceedings of the 19th Annual Review of Progress in Applied Computational Electromagnetics (ACES'2003), Monterey, CA, March 2003.

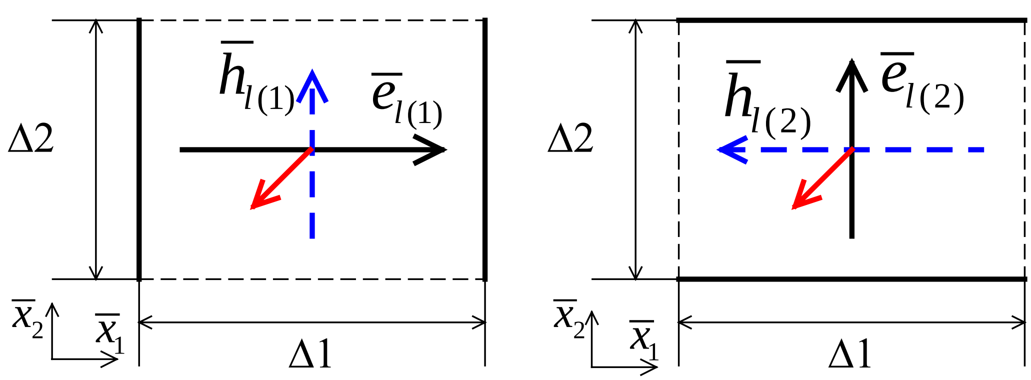

Fig 1. Two basis functions for a rectangular face. Black solid arrows are electric field components of the basis functions. Blue dashed arrows are magnetic field components of the basis functions. Red arrows are vectors normal to the face.

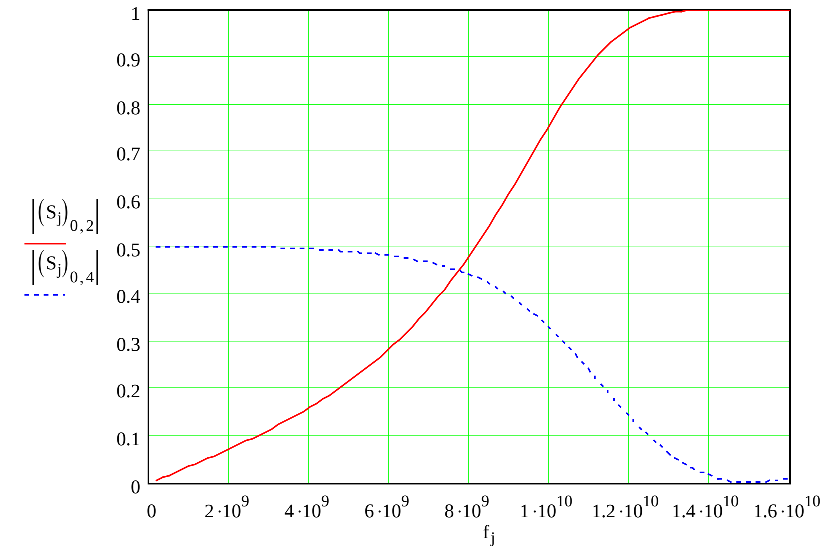

Fig. 2. Frequency dependencies of magnitudes of non-zero elements of scattering matrix of the cubic Trefftz finite element of Nikolskii.

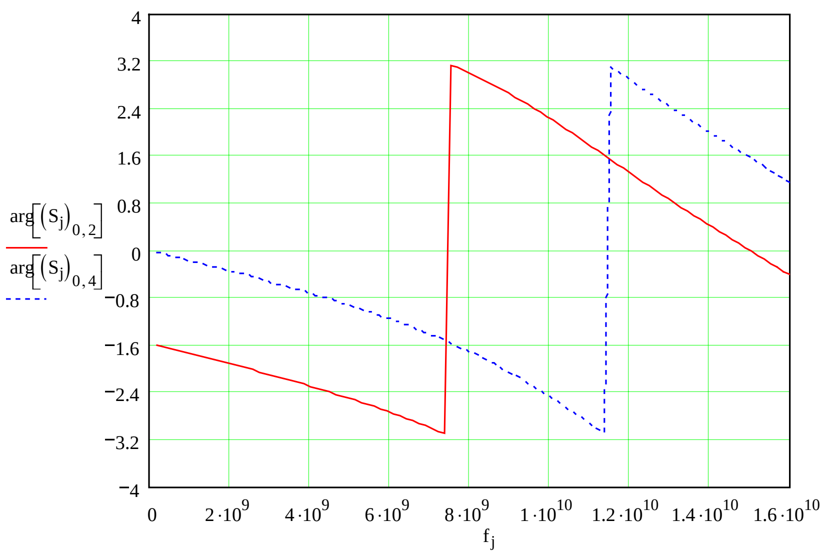

Fig. 3. Frequency dependencies of phases of non-zero elements of scattering matrix of the cubic Trefftz finite element of Nikolskii.

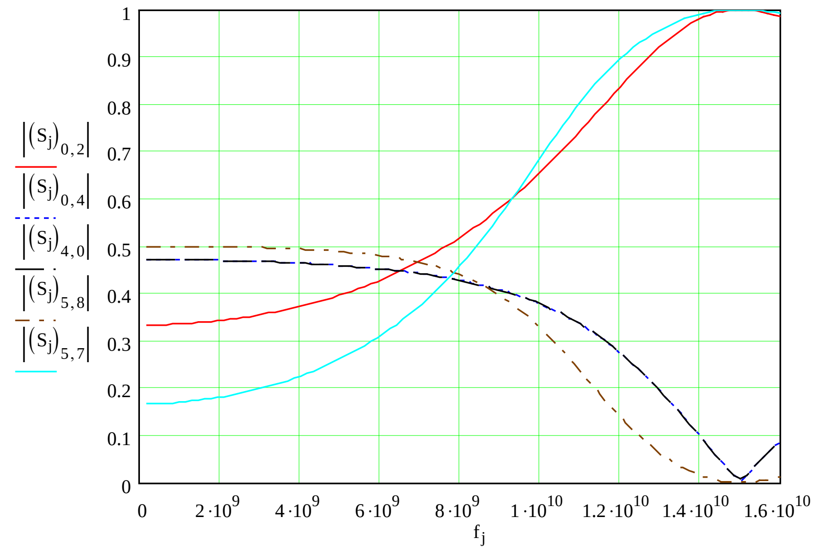

Fig. 4. Frequency dependencies of magnitudes of non-zero elements of scattering matrix of the brick Trefftz finite element of Nikolskii.

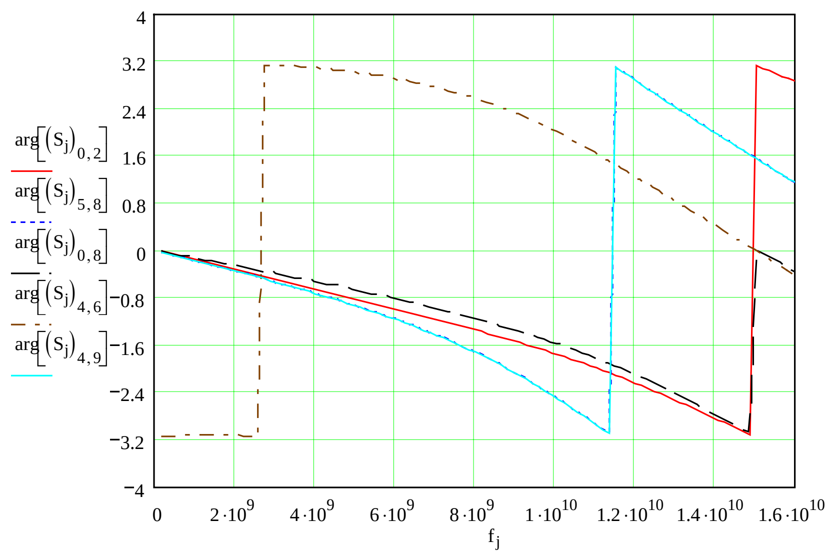

Fig. 5. Frequency dependencies of phases of non-zero elements of scattering matrix of the brick Trefftz finite element of Nikolskii.

